Non-Noether symmetries in singular dynamical systems

George Chavchanidze

Department of Theoretical Physics, A. Razmadze Institute of Mathematics,
1 Aleksidze Street, Tbilisi 0193, Georgia

Abstract. In the present paper geometric aspects of relationship between non-Noether symmetries and conservation laws in Hamiltonian systems is discussed. Case of irregular/constrained dynamical systems on presymplectic and Poisson manifolds is considered.

Keywords: Non-Noether symmetry; Conservation laws; Constrained dynamics;

MSC 2000: 70H33, 70H06, 53Z05

Reference: Georgian Math. J. 10 (2003) 057-061

1. Introduction

Noether's theorem associates conservation laws with particular continuous symmetries of the Lagrangian. According to the Hojman's theorem [1]-[3] there exists the definite correspondence between non-Noether symmetries and conserved quantities. In 1998 M. Lutzky showed that several integrals of motion might correspond to a single one-parameter group of non-Noether transformations [4]. In the present paper, the extension of Hojman-Lutzky theorem to singular dynamical systems is considered.

First of all let us recall some basic knowledge of description of the regular dynamical systems (see, e. g. [5]). In this case time evolution is governed by Hamilton's equation

$$i_{X_h}\omega + dh = 0, \tag{1}$$

where ω is the closed ($d\omega = 0$) and non-degenerate ($i_X\omega = 0 \Rightarrow X = 0$) 2-form, h is the Hamiltonian and $i_X\omega$ denotes contraction of X with ω. Since ω is non-degenerate, this gives rise to an isomorphism between the vector fields and 1-forms given by $i_X\omega + \alpha = 0$. The vector field is said to be Hamiltonian if it corresponds to exact form

$$i_{X_f}\omega + df = 0. \tag{2}$$

The Poisson bracket is defined as follows:

$$\{f , g\} = X_f\, g = -X_g\, f = i_{X_f}\, i_{X_g}\omega. \tag{3}$$

By introducing a bivector field W satisfying

$$i_Xi_Y\omega = i_W\, i_X\omega \wedge i_Y\omega, \tag{4}$$

Poisson bracket can be rewritten as

$$\{f , g\} = i_W\, df \wedge dg. \tag{5}$$

It's easy to show that

$$i_X i_Y L_Z \omega = i_{[Z,W]}\, i_X\omega \wedge i_Y\omega, \tag{6}$$

where the bracket $[\,\cdot\,,\,\cdot\,]$ is actually a supercommutator, for an arbitrary bivector field $W = \sum_s V^s \wedge U^s$ we have

$$[X,W] = \sum_s [X,V^s] \wedge U^s + \sum_s V^s \wedge [X,U^s] \tag{7}$$

Equation (6) is based on the following useful property of the Lie derivative

$$L_X i_W \omega = i_{[X,W]}\omega + i_W L_X \omega. \tag{8}$$

Indeed, for an arbitrary bivector field $W = \sum_s V^s \wedge U^s$ we have

$$L_X i_W \omega = L_X \sum_s i_{V^s \wedge U^s}\omega = L_X \sum_s i_{U^s} i_{V^s}\omega \tag{9}$$

$$= \sum_s i_{[X,U^s]} i_{V^s}\omega + \sum_s i_{U^s} i_{[X,V^s]}\omega + \sum_s i_{U^s} i_{V^s} L_X\omega = i_{[X,W]}\omega + i_W L_X\omega$$

where L_Z denotes the Lie derivative along the vector field Z. According to Liouville's theorem Hamiltonian vector field preserves ω

$$L_{X_f}\omega = 0; \tag{10}$$

therefore it commutes with W:

$$[X_f, W] = 0. \tag{11}$$

In the local coordinates z_s where $\omega = \sum_{rs} \omega^{rs} dz_r \wedge z_s$ bivector field W has the following form $W = \sum_{rs} W^{rs} \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}$ where W^{rs} is matrix inverted to ω^{rs}.

2. Case of regular Lagrangian systems

We can say that a group of transformations $g(z) = e^{zL_E}$ generated by the vector field E maps the space of solutions of equation onto itself if

$$i_{X_h} g_*(\omega) + g_*(dh) = 0 \tag{12}$$

For X_h satisfying

$$i_{X_h}\omega + dh = 0 \tag{13}$$

Hamilton's equation. It's easy to show that the vector field E should satisfy $[E, X_h] = 0$ Indeed,

$$i_{X_h} L_E \omega + dL_E h = L_E(i_{X_h}\omega + dh) = 0 \tag{14}$$

since $[E,X_h] = 0$. When E is not Hamiltonian, the group of transformations $g(z) = e^{zL_E}$ is non-Noether symmetry (in a sense that it maps solutions onto solutions but does not preserve action).

Theorem 1. (Lutzky, 1998) If the vector field E generates non-Noether symmetry, then the following functions are constant along solutions:

$$I^{(k)} = i_{W^k}\, \omega_E{}^k \qquad k = 1 \ldots n, \tag{15}$$

where W^k and $\omega_E{}^k$ are outer powers of W and $L_E\omega$.

Proof. We have to prove that $I^{(k)}$ is constant along the flow generated by the Hamiltonian. In other words, we should find that $L_{X_h} I^{(k)} = 0$ is fulfilled. Let us consider $L_{X_h} I^{(1)}$

$$L_{X_h} I^{(1)} = L_{X_h}(i_W \omega_E) = i_{[X_h\,,\,W]}\omega_E + i_W L_{X_h}\omega_E, \tag{16}$$

where according to Liouville's theorem both terms $[X_h\,,W] = 0$ and

$$i_W L_{X_h} L_E \omega = i_W L_E L_{X_h} \omega = 0 \tag{17}$$

since $[E\ ,\ X_h] = 0$ and $L_{X_h}\omega = 0$ vanish. In the same manner one can verify that $L_{X_h} I^{(k)} = 0$

Remark 1. Theorem is valid for a larger class of generators E . Namely, if $[E\,,X_h] = X_f$ where X_f is an arbitrary Hamiltonian vector field, then $I^{(k)}$ is still conserved. Such a symmetries map the solutions of the equation $i_{X_h}\omega + dh = 0$ on solutions of

$$i_{X_h} g_*(\omega) + d(g_* h + f) = 0 \tag{18}$$

Remark 2. Discrete non-Noether symmetries give rise to the conservation of $I^{(k)} = i_{W^k} g_*(\omega)^k$ where $g_*(\omega)$ is transformed ω.

Remark 3. If $I^{(k)}$ is a set of conserved quantities associated with E and f is any conserved quantity, then the set of functions $\{I^{(k)}\ ,\ f\}$ (which due to the Poisson theorem are integrals of motion) is associated with $[X_h\ ,\ E]$. Namely it is easy to show by taking the Lie derivative of (15) along vector field E that

$$\{I^{(k)}\,,\,f\} = i_{W^k}\omega^k{}_{[X_f\,,\,E]} \tag{19}$$

is fulfilled. As a result conserved quantities associated with Non-Noether symmetries form Lie algebra under the Poisson bracket.

Remark 4. If generator of symmetry satisfies Yang-Baxter equation $[[E[E\,,W]]W] = 0$ Lutzky's conservation laws are in involution [7] $\{Y^{(l)}\,,\,Y^{(k)}\} = 0$

3. Case of irregular Lagrangian systems

The singular Lagrangian (Lagrangian with vanishing Hessian) leads to degenerate 2-form ω and we no longer have isomorphism between vector fields and 1-forms. Since there exists a set of "null vectors" u_s such that $i_{u_s}\omega = 0 \qquad s = 1,2 \dots n - \mathrm{rank}(\omega)$, every Hamiltonian vector field is defined up to linear combination of vectors u_s. By identifying X_f with $X_f + \sum_s C_s u_s$, we can introduce equivalence class X_f^* (then all u_s belong to 0^*). The bivector field W is also far from being unique, but if W_1 and W_2 both satisfy

$$i_X i_Y\, \omega = i_{W_{1,2}}\, i_X\omega \wedge i_Y\omega, \tag{20}$$

then

$$i_{(W_1 - W_2)}\, i_X\omega \wedge i_Y\omega = 0 \qquad \forall X,Y \tag{21}$$

is fulfilled. It is possible only when

$$W_1 - W_2 = \sum_s v_s \wedge u_s \tag{22}$$

where v_s are some vector fields and $i_{u_s}\omega = 0$ (in other words when $W_1 - W_2$ belongs to the class 0^*)

Theorem 2. If the non-Hamiltonian vector field E satisfies $[E\, ,\, X_h^*] = 0^*$ commutation relation (generates non-Noether symmetry), then the functions

$$I^{(k)} = i_{W^k}\omega_E{}^k \qquad k = 1 \dots \mathrm{rank}(\omega) \tag{23}$$

(where $\omega_E = L_E\omega$) are constant along trajectories.

Proof. Let's consider $I^{(1)}$

$$L_{X_h{}^*}I^{(1)} = L_{X_h{}^*}(i_W\omega_E) = i_{[X_h{}^*\, ,\, W]}\omega_E + i_W L_{X_h{}^*}\omega_E = 0 \tag{24}$$

The second term vanishes since $[E\, ,\, X_h^*] = 0^*$ and $L_{X_h{}^*}\omega = 0$. The first one is zero as far as $[X_h^*\, ,\, W^*] = 0^*$ and $[E\, ,\, 0^*] = 0^*$ are satisfied. So $I^{(1)}$ is conserved. Similarly one can show that $L_{X_h}I^{(k)} = 0$ is fulfilled.

Remark 5. W is not unique, but $I^{(k)}$ doesn't depend on choosing representative from the class W^*.

Remark 6. Theorem is also valid for generators E satisfying $[E\, ,\, X_h^*] = X_f^*$

Example 1. Hamiltonian description of the relativistic particle leads to the following action

$$A = \int p_0 dx_0 + \sum_s p_s dx_s \tag{25}$$

where $p_0 = (p^2 + m^2)^{1/2}$ with vanishing canonical Hamiltonian and degenerate 2-form defined by

$$p_0\omega = \sum_s (p_s dp_s \wedge dx_0 + p_0 dp_s \wedge dx_s). \tag{26}$$

ω possesses the "null vector field" $i_u\omega = 0$

$$u = p_0\frac{\partial}{\partial x_0} + \sum_s p_s\frac{\partial}{\partial x_s}. \tag{27}$$

One can check that the following non- Hamiltonian vector field

$$E = p_0 x_0\frac{\partial}{\partial x_0} + p_1 x_1\frac{\partial}{\partial x_1} + \ldots + p_n x_n\frac{\partial}{\partial x_n} \tag{28}$$

generates non-Noether symmetry. Indeed, E satisfies $[E\,,\, X_h^*] = 0^*$ because of $X_h^* = 0^*$ and $[E,u] = u$. Corresponding integrals of motion are combinations of momenta:

$$I^{(1)} = \sum_s p_s \tag{29}$$

$$I^{(2)} = \sum_{r>s} p_r p_s$$

$$\ldots$$

$$I^{(n)} = \prod_s p_s$$

This example shows that the set of conserved quantities can be obtained from a single one-parameter group of non-Noether transformations.

Acknowledgements. Author is grateful to Z. Giunashvili and M. Maziashvili for constructive discussions and particularly grateful to George Jorjadze for invaluable help. This work was supported by INTAS (00-00561) and Scholarship from World Federation of Scientists.

References

[1] S. Hojman, *A new conservation law constructed without using either Lagrangians or Hamiltonians,* J. Phys. A: Math. Gen. 25 L291-295, 1992

[2] F. González-Gascón, *Geometric foundations of a new conservation law discovered by Hojman,* J. Phys. A: Math. Gen. 27 L59-60, 1994

[3] M. Lutzky, *Remarks on a recent theorem about conserved quantities,* J. Phys. A: Math. Gen. 28 L637-638, 1995

[4] M. Lutzky, *New derivation of a conserved quantity for Lagrangian systems,* J. Phys. A: Math. Gen. 15 L721-722, 1998

[5] N.M.J. Woodhouse, *Geometric Quantization*, Claredon, Oxford, 1992.
[6] G. Chavchanidze, *Bi-Hamiltonian structure as a shadow of non-Noether symmetry*, math-ph/0106018, 2001