Non-Noether symmetries in singular dynamical systems

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Abstract. In the present paper geometric aspects of relationship between non-Noether symmetries and conservation laws in Hamiltonian systems is discussed. Case of irregular/constrained dynamical systems on presymplectic and Poisson manifolds is considered.

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1. Introduction

Noether's theorem associates conservation laws with particular continuous symmetries of the Lagrangian. According to the Hojman's theorem [1]-[3] there exists the definite correspondence between non-Noether symmetries and conserved quantities. In 1998 M. Lutzky showed that several integrals of motion might correspond to a single one-parameter group of non-Noether transformations [4]. In the present paper, the extension of Hojman-Lutzky theorem to singular dynamical systems is considered.

First of all let us recall some basic knowledge of description of the regular dynamical systems (see, e. g. [5]). In this case time evolution is governed by Hamilton's equation
\[ i_{\dot{x}}\omega + dh = 0, \]
where \( \omega \) is the closed (\( d\omega = 0 \)) and non-degenerate (\( i_{\dot{x}}\omega = 0 \Rightarrow \dot{x} = 0 \)) 2-form, \( h \) is the Hamiltonian and \( i_{\dot{x}}\omega \) denotes contraction of \( \dot{x} \) with \( \omega \). Since \( \omega \) is non-degenerate, this gives rise to an isomorphism between the vector fields and 1-forms given by \( i_{\dot{x}}\omega + \alpha = 0 \). The vector field is said to be Hamiltonian if it corresponds to exact form
\[ i_{\dot{x}}\omega + df = 0. \]

The Poisson bracket is defined as follows:
\[ \{f, g\} = X_f g = -X_g f = i_{\dot{x}}\omega. \]

By introducing a bivector field \( W \) satisfying
\[ i_{\dot{x}}i_{\dot{y}}\omega = i_W i_{\dot{x}}\omega \wedge i_{\dot{y}}\omega, \]
Poisson bracket can be rewritten as
\[ \{f, g\} = i_W df \wedge dg. \]
It's easy to show that
\[ i_X i_Y L_Z \omega = i_{[Z,W]} i_X \omega \wedge i_Y \omega , \tag{6} \]
where the bracket \([ \cdot , \cdot ]\) is actually a supercommutator, for an arbitrary bivector field \( W = \sum_s V^s \wedge U^s \) we have
\[ [X,W] = \sum_s [X,V^s] \wedge U^s + \sum_s V^s \wedge [X,U^s] \tag{7} \]
Equation (6) is based on the following useful property of the Lie derivative
\[ L_X i_W \omega = i_{[X,W]} \omega + i_W L_X \omega . \tag{8} \]
Indeed, for an arbitrary bivector field \( W = \sum_s V^s \wedge U^s \) we have
\[
\begin{align*}
L_X i_W \omega &= L_X \sum_s i_{V^s} \wedge U^s \omega = L_X \sum_s i_{U^s} i_{V^s} \omega \\
&= \sum_s i_{[X,U^s]} i_{V^s} \omega + \sum_s i_{U^s} i_{[X,V^s]} \omega + \sum_s i_{U^s} i_{V^s} L_X \omega = i_{[X,W]} \omega + i_W L_X \omega
\end{align*}
\]
where \( L_Z \) denotes the Lie derivative along the vector field \( Z \). According to Liouville's theorem Hamiltonian vector field preserves \( \omega \)
\[ L_{X_f} \omega = 0; \tag{10} \]
therefore it commutes with \( W \):
\[ [X_f,W] = 0. \tag{11} \]
In the local coordinates \( z_s \) where \( \omega = \sum_{rs} \omega^{rs} dz_r \wedge z_s \) bivector field \( W \) has the following form
\[ W = \sum_{rs} W^{rs} \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s} \] where \( W^{rs} \) is matrix inverted to \( \omega^{rs} \).

2. Case of regular Lagrangian systems

We can say that a group of transformations \( g(z) = e^{z L_E} \) generated by the vector field \( E \) maps the space of solutions of equation onto itself if
\[ i_{X_h} g_*(\omega) + g_*(dh) = 0 \tag{12} \]
For \( X_h \) satisfying
\[ i_{X_h} \omega + dh = 0 \tag{13} \]
Hamilton's equation. It's easy to show that the vector field \( E \) should satisfy
\[ [E , X_h] = 0 \] Indeed,
\[ i_{X_h} L_E \omega + dL_E h = L_E (i_{X_h} \omega + dh) = 0 \tag{14} \]
since $[E,X_h] = 0$. When $E$ is not Hamiltonian, the group of transformations $g(z) = e^{2L_E}$ is non-Noether symmetry (in a sense that it maps solutions onto solutions but does not preserve action).

Theorem 1. (Lutzky, 1998) If the vector field $E$ generates non-Noether symmetry, then the following functions are constant along solutions:

$$I^{(k)} = i_W^k \omega^k_{E} \quad k = 1...n,$$

(15)

where $W^k$ and $\omega^k_E$ are outer powers of $W$ and $L_E \omega$.

Proof. We have to prove that $I^{(k)}$ is constant along the flow generated by the Hamiltonian. In other words, we should find that $L_{X_h} I^{(k)} = 0$ is fulfilled. Let us consider $L_{X_h} I^{(1)}$

$$L_{X_h} I^{(1)} = L_{X_h} (i_W \omega_E) = i_{[X_h, W]} \omega_E + i_W L_{X_h} \omega_E,$$

where according to Liouville's theorem both terms $[X_h, W] = 0$ and

$$i_W L_{X_h} L_E \omega = i_W L_E L_{X_h} \omega = 0$$

(17)

since $[E, X_h] = 0$ and $L_{X_h} \omega = 0$ vanish. In the same manner one can verify that $L_{X_h} I^{(k)} = 0$

Remark 1. Theorem is valid for a larger class of generators $E$. Namely, if $[E, X_h] = X_f$ where $X_f$ is an arbitrary Hamiltonian vector field, then $I^{(k)}$ is still conserved. Such a symmetries map the solutions of the equation $i_{X_h} \omega + dh = 0$ on solutions of

$$i_{X_h} g_\ast(\omega) + d(g_\ast h + f) = 0$$

(18)

Remark 2. Discrete non-Noether symmetries give rise to the conservation of $I^{(k)} = i_W^k g_\ast(\omega)^k$ where $g_\ast(\omega)$ is transformed $\omega$.

Remark 3. If $I^{(k)}$ is a set of conserved quantities associated with $E$ and $f$ is any conserved quantity, then the set of functions $\{I^{(k)}, f\}$ (which due to the Poisson theorem are integrals of motion) is associated with $[X_h, E]$. Namely it is easy to show by taking the Lie derivative of (15) along vector field $E$ that

$$\{I^{(k)}, f\} = i_W^k \omega^k_{[X_f, E]}$$

(19)

is fulfilled. As a result conserved quantities associated with Non-Noether symmetries form Lie algebra under the Poisson bracket.

Remark 4. If generator of symmetry satisfies Yang-Baxter equation $[[E, W]]W = 0$ Lutzky's conservation laws are in involution $[7] \{Y^{(l)}, Y^{(k)}\} = 0$

3. Case of irregular Lagrangian systems
The singular Lagrangian (Lagrangian with vanishing Hessian) leads to degenerate 2-form $\omega$ and we no longer have isomorphism between vector fields and 1-forms. Since there exists a set of "null vectors" $u_s$ such that $i_{u_s}\omega = 0 \quad s = 1, 2 \ldots n - \text{rank}(\omega)$, every Hamiltonian vector field is defined up to linear combination of vectors $u_s$. By identifying $X_f$ with $X_f + \sum C_s u_s$, we can introduce equivalence class $X_f^*$ (then all $u_s$ belong to $0^*$). The bivector field $W$ is also far from being unique, but if $W_1$ and $W_2$ both satisfy

$$i_X i_Y \omega = i_{W_{1,2}} i_X \omega \wedge i_Y \omega,$$

then

$$i_{(W_1 - W_2)} i_X \omega \wedge i_Y \omega = 0 \quad \forall X, Y$$

is fulfilled. It is possible only when

$$W_1 - W_2 = \sum v_s \wedge u_s$$

where $v_s$ are some vector fields and $i_{u_s}\omega = 0$ (in other words when $W_1 - W_2$ belongs to the class $0^*$)

**Theorem 2.** If the non-Hamiltonian vector field $E$ satisfies $[E, X_h^*] = 0^*$ commutation relation (generates non-Noether symmetry), then the functions

$$I^{(k)} = i_{W^k} \omega_E^k \quad k = 1 \ldots \text{rank}(\omega)$$

(where $\omega_E = L_E \omega$) are constant along trajectories.

**Proof.** Let's consider $I^{(1)}$

$$L_{X_h} I^{(1)} = L_{X_h} (i_{W} \omega_E) = i_{[X_h^*, W]} \omega_E + i_{W} L_{X_h} \omega_E = 0$$

The second term vanishes since $[E, X_h^*] = 0^*$ and $L_{X_h} \omega = 0$. The first one is zero as far as $[X_h^*, W^*] = 0^*$ and $[E, 0^*] = 0^*$ are satisfied. So $I^{(1)}$ is conserved. Similarly one can show that $L_{X_h} I^{(k)} = 0$ is fulfilled.

**Remark 5.** $W$ is not unique, but $I^{(k)}$ doesn't depend on choosing representative from the class $W^*$.

**Remark 6.** Theorem is also valid for generators $E$ satisfying $[E, X_h^*] = X_f^*$

**Example 1.** Hamiltonian description of the relativistic particle leads to the following action
\[ A = \int p_0 dx_0 + \sum_s p_s dx_s \]  

where \( p_0 = (p^2 + m^2)^{1/2} \) with vanishing canonical Hamiltonian and degenerate 2-form defined by

\[ p_0 \omega = \sum_s (p_s dp_s \wedge dx_0 + p_0 dp_s \wedge dx_s). \]

\( \omega \) possesses the "null vector field" \( i_u \omega = 0 \)

\[ u = p_0 \frac{\partial}{\partial x_0} + \sum_s p_s \frac{\partial}{\partial x_s}. \]

One can check that the following non-Hamiltonian vector field

\[ E = p_0 x_0 \frac{\partial}{\partial x_0} + p_1 x_1 \frac{\partial}{\partial x_1} + \cdots + p_n x_n \frac{\partial}{\partial x_n} \]

generates non-Noether symmetry. Indeed, \( E \) satisfies \( [E, X_h^*] = 0^* \) because of \( X_h^* = 0^* \) and \( [E,u] = u \). Corresponding integrals of motion are combinations of momenta:

\[ I^{(1)} = \sum_s p_s \]

\[ I^{(2)} = \sum_{r > s} p_r p_s \]

\[ \cdots \]

\[ I^{(n)} = \prod_s p_s \]

This example shows that the set of conserved quantities can be obtained from a single one-parameter group of non-Noether transformations.

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References


