Non-Noether symmetry of the modified Boussinesq equations

George Chavchanidze

Department of Theoretical Physics, A. Razmadze Institute of Mathematics, 1 Aleksidze Street, Tbilisi 0193, Georgia

Abstract. We investigate one-parameter non-Noether symmetry group of the modified Boussinesq equations and show that this symmetry naturally yields infinite sequence of conservation laws.

Keywords: Non-Noether symmetry; Conservation laws; Modified Boussinesq system;

MSC 2000: 70H33; 70H06; 58J70; 53Z05; 35A30

In Hamiltonian systems, conservation laws are closely related to symmetries of evolutionary equations. In case of modified Boussinesq hierarchy this relationship is especially tight as its entire infinite set of conservation laws forms a single involutive orbit of a simple one-parameter symmetry group. We discuss some geometric properties of this symmetry and show how its properties ensure involutivity of conservation laws.

Recall that the modified Boussinesq system is formed by the following set of partial differential equations

$$\begin{aligned} u_t &= cv_{xx} + u_x v + uv_x \\ v_t &= -cu_{xx} + uu_x + kvv_x \end{aligned} \tag{1}$$

where $u = u(x, t)$, $v = v(x, t)$ are smooth functions on R^2 subjected to zero boundary conditions $u(\pm\infty, t) = v(\pm\infty, t) = 0$, while c and k are some real constants. In cases $k = -1$ and $k = 3$ modified Boussinesq system has non-trivial bi-Hamiltonian structure that drastically simplifies analysis of the system in these sectors. The first case is described in [2],[5],[6], while in the present paper we focus on the second sector and show that in case $k = 3$ bi-Hamiltonian structure of modified Boussinesq system is related to non-Noether symmetry [1] of equations (1). Thus in case $k = 3$ modified Boussinesq equations

$$\begin{aligned} u_t &= cv_{xx} + u_x v + uv_x \\ v_t &= -cu_{xx} + uu_x + 3vv_x \end{aligned} \tag{2}$$

can be rewritten in bi-Hamiltonian form

$$\begin{aligned} u_t &= W(dh \wedge du) = \hat{W}(d\hat{h} \wedge du) \\ v_t &= W(dh \wedge dv) = \hat{W}(d\hat{h} \wedge dv) \end{aligned} \tag{3}$$

where W and $\hat{W}$ are compatible Poison bivector fields, i.e.

$$[W , W] = [W , \hat{W}] = [\hat{W} , \hat{W}] = 0 \quad (4)$$

defined as follows

$$W = \int_{-\infty}^{+\infty} \tfrac{1}{2}(A \wedge A_x + B \wedge B_x)dx \quad (5)$$

$$\hat{W} = \int_{-\infty}^{+\infty} (uB \wedge A_x + vB \wedge B_x - cA_x \wedge B_x)dx$$

Note that A, B are vector fields that for every smooth functional R = R(u) are defined via variational derivatives

$$A(R) = \frac{\delta R}{\delta u}, \qquad B(R) = \frac{\delta R}{\delta v}. \quad (6)$$

Corresponding Hamiltonians in bi-Hamiltonian realization (3) are

$$h = \tfrac{1}{2}\int_{-\infty}^{+\infty} (u^2 v + v^3 + 2cuv_x)dx \quad (7)$$

$$\hat{h} = \tfrac{1}{2}\int_{-\infty}^{+\infty} (u^2 + v^2)dx$$

This bi-Hamiltonian structure is related to symmetry of equations (2), but before we proceed let us remind that symmetry of evolutionary equations is given by the group of transformations

$$(u , v) \text{ ? } (g(u) , g(v)) \quad (8)$$

which commutes with time evolution

$$\frac{d}{dt}g(u) = g(u_t), \qquad \frac{d}{dt}g(v) = g(v_t) \quad (9)$$

In case of continuous one-parameter groups of transformation

$$g(u) = e^{zL_E}(u) = u + zL_E u + \tfrac{1}{2}z^2(L_E)^2 u + \ldots \quad (10)$$

$$g(v) = e^{zL_E}(v) = v + zL_E v + \tfrac{1}{2}z^2(L_E)^2 v + \ldots$$

generated by some vector field E, relation (9) gives rise to the following conditions for the generator of symmetry E

$$E(u)_t = cE(v)_{xx} + E(u)_x v + uE(v)_x + u_x E(v) + E(u)v_x \quad (11)$$

$$E(v)_t = - cE(u)_{xx} + uE(u)_x + 3vE(v)_x + E(u)u_x + 3E(v)v_x$$

Among solutions of equations (11) there is one important vector field — the generator of non-Noether symmetry which has the following form

$$E = \int_{-\infty}^{+\infty} \{[xuv + 2t(u^3 + 3uv^2 + 6cvv_x - 2c^2u_{xx})]A_x - cxvA_{xx} + (xuu_x + xvv_x)B + [xu^2 + 2xv^2 + 2t(5v^3 + 3u^2v - 6cvu_x - 2c^2v_{xx})]B_x + cxuB_{xx}\}dx \qquad (12)$$

Applying one-parameter group of transformations

$$g(z) = e^{zL_E} \qquad (13)$$

generated by the vector field E to the centre of Poisson algebra which in our case is formed by functional

$$J = \int_{-\infty}^{+\infty} (ku + mv)dx \qquad (14)$$

where k, m are arbitrary constants, produces one-parameter family of functions

$$J(z) = e^{zL_E}J = J + zL_EJ + ½(zL_E)^2J + \dots \qquad (15)$$

(actually this is the orbit of non-Noether symmetry group that passes centre of Poisson algebra). It is interesting that the functionals $(L_E)^mJ$ are in involution.

Theorem 1. The orbit (15) of the non-Noether symmetry group generated by the vector field (12) is involutive

$$\{J(x) , J(y)\} = 0 \qquad \forall x, y \in R \qquad (16)$$

and the functionals

$$J^{(m)} = (L_E)^mJ \qquad (17)$$

form Lenard scheme with respect to bi-Hamiltonian structure (5) and produce involutive sequence of conservation laws of the modified Boussinesq hierarchy.

Proof. The theorem follows from simple geometric properties of the vector field E. In particular taking the Lie derivative of Poisson bivector field W along E one gets the second Poisson bivector involved in bi-Hamiltonian system (5)

$$\hat{W} = [E , W] \qquad (18)$$

while the Lie derivative of $\hat{W}$ along E vanishes $[E , \hat{W}] = 0$ These properties ensure that the functionals (17) are in involution (the Poisson bracket of arbitrary two conservation laws from infinite family (17) vanishes)

$$\{J^{(k)} , J^{(m)}\} = 0 \qquad k, m = 0, 1, 2 \dots \qquad (19)$$

Indeed, by applying m-th order Lie derivative $(L_E)^m$ to the relation

$$W(dJ^{(0)}) = 0 \qquad (20)$$

which reflects the fact that $J^{(0)}$ belongs to the centre of Poisson algebra, its easy to prove that the functionals (17) form Lenard scheme

$$W(dJ^{(m+1)}) = -(1+m)[E, W](dJ^{(m)}) \tag{21}$$

with respect to bi-Hamiltonian system (5) From the other hand it is well known [4] that functionals involved in Lenard scheme are in involution. In the same time calculating the functional

$$J^{(2)} = (L_E)^2 J^{(0)} = m\int_{-\infty}^{+\infty} (u^2 v + v^3 + 2cuv_x)dx = 2mH \tag{22}$$

gives rise to Hamiltonian of the modified Boussinesq system and functionals $J^{(m)}$ being in involution with Hamiltonian must be conservation laws.

By calculating Lie derivatives of $J^{(0)}$ along the vector field E one can get explicit form of the conservation laws of the modified Boussinesq system:

$$\begin{aligned}
&J^{(0)} = \int_{-\infty}^{+\infty} (ku + mv)dx \\
&J^{(1)} = L_E J^{(0)} = \frac{m}{2}\int_{-\infty}^{+\infty} (u^2 + v^2)dx \\
&J^{(2)} = (L_E)^2 J^{(0)} = m\int_{-\infty}^{+\infty} (u^2 v + v^3 + 2cuv_x)dx \\
&J^{(3)} = (L_E)^3 J^{(0)} = \frac{3m}{4}\int_{-\infty}^{+\infty} (u^4 + 5v^4 + 6u^2v^2 \\
&- 12cv^2u_x + 4c^2u_x{}^2 + 4c^2v_x{}^2)dx \\
&J^{(m)} = (L_E)^m J^{(0)} = L_E J^{(m-1)}
\end{aligned} \tag{23}$$

The fact that the infinite sequence of conservation laws of modified Boussinesq hierarchy form single orbit of the one-parameter non-Noether symmetry group indicates that non-Noether symmetries may play an important role in analysis of certain integrable models where they drastically simplify calculation of conservation laws and shed more light on geometric origin of integrable hierarchies. Basic results of the paper can be extended to the case of periodic boundary conditions $u(-\infty) = u(+\infty)$ and $v(-\infty) = v(+\infty)$ when the modified Boussinesq equations can be considered as bi-Hamiltonian system on a loop space [4]. Note however that in the periodic case the symmetry (12) does not seem to preserve boundary conditions.

Acknowledgements. The research described in this publication was made possible in part by Award No. GEP1-3327-TB-03 of the Georgian Research and Development

Foundation (GRDF) and the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

References

[1] G. Chavchanidze, *Non-Noether symmetries and their influence on phase space geometry,* J. Geom. Phys. 48 190-202, 2003

[2] A. Fordy and J. Gibbons, *Factorization of operators. II,* J. Math. Phys. 22, No. 6, 1170–1175, 1981

[3] F. Magri, *A simple model of the integrable Hamiltonian equation,* J. Math. Phys. 19 No. 5, 1156-1162, 1978

[4] O. Mokhov, *Symplectic and Poisson Geometry on Loop Spaces of Smooth Manifolds and Integrable Equations,* Routledge, 2004

[5] J.P. Wang, *A list of 1 + 1 dimensional integrable equations and their properties,* J. Nonlinear Math. Phys. 9, suppl. 1, 213–233, 2002

[6] J.P. Wang, *Symmetries and conservation laws of evolution equations,* Ph.D. Thesis, Vrije Universiteit van Amsterdam, 1998