Bi-Hamiltonian structure as a shadow of non-Noether symmetry

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Abstract. In the present paper correspondence between non-Noether symmetries and bi-Hamiltonian structures is disscussed. We show that in regular Hamiltonian systems presence of the global bi-Hamiltonian structure is caused by symmetry of the space of solution. As an example well known bi-Hamiltonian realisation of Korteweg-De Vries equation is disscussed.

Keywords: Bi-Hamiltonian system; Non-Noether symmetry; Non-Cartan symmetry; Korteweg- De Vries equation.

MSC 2000: 70H33, 70H06, 53Z05

Reference: Georgian Math. J. 10 (2003) 057-061

Noether theorem, Lutzky's theorem, bi-Hamiltonian formalism and bidifferential calculi are often used in generating conservation laws and all this approaches are unified by the single idea — to construct conserved quantities out of some invariant geometric object (generator of the symmetry — Hamiltonian vector field in Noether theorem, non-Hamiltonian one in Lutzky's approach, closed 2-form in bi-Hamiltonian formalism and auxiliary differential in case of bidifferential calculi). There is close relationship between later three approaches. Some aspects of this relationship has been uncovered in [3],[4]. In the present paper it is discussed how bi-Hamiltonian structure can be interpreted as a manifestation of symmetry of space of solutions. Good candidate for this role is non-Noether symmetry. Such a symmetry is a group of transformation that maps the space of solutions of equations of motion onto itself, but unlike the Noether one, does not preserve action.

In the case of regular Hamiltonian system phase space is equipped with symplectic form ω (closed $d\omega = 0$ and nondegenerate $i_X\omega = 0 \Rightarrow X = 0$ 2-form) and time evolution is governed by Hamilton's equation

$$i_{X_h}\omega + dh = 0 \tag{1}$$

where X_h is Hamiltonian vector field that defines time evolution

$$\frac{df}{dt} = X_h(f) \tag{2}$$

for any function f and $i_{X_h}\omega$ denotes contraction of X_h and ω. Vector field is said to be (locally) Hamiltonian if it preserves ω. According to the Liouville's theorem X_h defined by (1) automatically preserves ω due to relation

$$L_{X_h}\omega = di_{X_h}\omega + i_{X_h}d\omega = -\,ddh = 0 \tag{3}$$

One can show that group of transformations of phase space generated by any non-Hamiltonian vector field E

$$g(a) = e^{aL_E} \tag{4}$$

does not preserve action

$$g_*(A) = g_*(\int pdq - hdt) = \int g_*(pdq - hdt) \neq 0 \tag{5}$$

because $d(L_E(pdq - hdt)) = L_E\omega - dE(h) \wedge dt \neq 0$ (first term in r.h.s. does not vanish since E is non-Hamiltonian and as far as E is time independent $L_E\omega$ and $dE(h) \wedge dt$ are linearly independent 2-forms). As a result every non-Hamiltonian vector field E commuting with X_h leads to the non-Noether symmetry (since E preserves vector field tangent to solutions $L_E(X_h) = [E\ ,\ X_h] = 0$ it maps the space of solutions onto itself). Any such symmetry yields the following integrals of motion [1],[2],[4],[5]

$$I^{(k)} = Tr(R^k) \qquad k = 1,2 \ldots n \tag{6}$$

where $R = \omega^{-1}L_E\omega$ and n is half-dimension of phase space.

It is interesting that for any non-Noether symmetry, triple $(h,\ \omega,\ \omega_E)$ carries bi-Hamiltonian structure (§4.12 in [6],[7]-[9]). Indeed ω_E is closed ($d\omega_E = dL_E\omega = L_Ed\omega = 0$) and invariant ($L_{X_h}\omega_E = L_{X_h}L_E\omega = L_EL_{X_h}\omega = 0$) 2-form (but generic ω_E is degenerate). So every non-Noether symmetry quite naturally endows dynamical system with bi-Hamiltonian structure.

Now let's discuss how non-Noether symmetry can be recovered from bi-Hamiltonian system. Generic bi-Hamiltonian structure on phase space consists of Hamiltonian system h, ω and auxiliary closed 2- form ω^* satisfying $L_{X_h}\omega^* = 0$. Let us call it global bi-Hamiltonian structure whenever ω^* is exact (there exists 1-form θ^* such that $\omega^* = d\theta^*$) and X_h is (globally) Hamiltonian vector field with respect to ω^* ($i_{X_h}\omega^* + dh^* = 0$). As far as ω is nondegenerate there exists vector field E^* such that $i_{E^*}\omega = \theta^*$. By construction

$$L_{E^*}\omega = \omega^* \tag{7}$$

Indeed

$$L_{E^*}\omega = di_{E^*}\omega + i_{E^*}d\omega = d\theta^* = \omega^* \tag{8}$$

And

$$i_{[E^*,X_h]}\omega = L_{E^*}(i_{X_h}\omega) - i_{X_h}L_{E^*}\omega = -\,d(E^*(h) - h^*) = -\,dh' \tag{9}$$

In other words $[X_h , E^*]$ is Hamiltonian vector field, i. e., $[X_h , E] = X_{h'}$. So E^* is not generator of symmetry since it does not commute with X_h but one can construct (locally) Hamiltonian counterpart of E^* (note that E^* itself is non-Hamiltonian) — X_g with

$$g(z) = \int_0^t h' d\tau \tag{10}$$

Here integration along solution of Hamilton's equation, with fixed origin and end point in z(t) = z, is assumed. Note that (10) defines g(z) only locally and, as a result, X_g is a locally Hamiltonian vector field, satisfying, by construction, the same commutation relations as E^* (namely $[X_h , X_g] = X_{h'}$). Finally one recovers generator of non-Noether symmetry — non-Hamiltonian vector field $E = E^* - X_g$ commuting with X_h and satisfying

$$L_E\omega = L_{E^*}\omega - L_{X_g}\omega = L_{E^*}\omega = \omega^* \tag{11}$$

(thanks to Liouville's theorem $L_{X_g}\omega = 0$). So in case of regular Hamiltonian system every global bi-Hamiltonian structure is naturally associated with (non-Noether) symmetry of space of solutions.

Example 1. As a toy example one can consider free particle

$$h = ½ \sum_m p_m^2 \qquad \omega = \sum_m dp_m \wedge dq_m \tag{12}$$

this Hamiltonian system can be extended to the bi-Hamiltonian one

$$h, \omega, \omega^* = \sum_m p_m dp_m \wedge dq_m \tag{13}$$

clearly $d\omega^* = 0$ and X_h preserves ω^*. Conserved quantities p_m are associated with this simple bi-Hamiltonian structure. This system can be obtained from the following (non-Noether) symmetry (infinitesimal form)

$$\begin{aligned} q_m &\rightarrow (1 + ap_m)q_m \\ p_m &\rightarrow (1 + ap_m)p_m \end{aligned} \tag{14}$$

Example 2. The earliest and probably the most well known bi-Hamiltonian structure is the one discovered by F. Magri and assosiated with Korteweg- De Vries integrable hierarchy. The KdV equation

$$u_t + u_{xxx} + uu_x = 0 \tag{15}$$

(zero boundary conditions for u and its derivatives are assumed) appears to be Hamilton's equation

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$$i_{X_h}\omega + dh = 0 \tag{16}$$

where

$$X_h = \int_{-\infty}^{+\infty} dx\, u_t \frac{\delta}{\delta u} \tag{17}$$

(here $\frac{\delta}{\delta u}$ denotes variational derivative with respect to the field u(x)) is the vector field tangent to the solutions,

$$\omega = \int_{-\infty}^{+\infty} dx\, du \wedge dv \tag{18}$$

is the symplectic form (here v is defined by $v_x = u$) and the function

$$h = \int_{-\infty}^{+\infty} dx \left(\frac{u^3}{3} - u_x^{\ 2} \right) \tag{19}$$

plays the role of Hamiltonian. This dynamical system possesses non-trivial symmetry — one-parameter group of non-cannonical transformations $g(a) = e^{L_E}$ generated by the non-Hamiltonian vector field

$$E = \int_{-\infty}^{+\infty} dx \left(u_{xx} + \frac{u^2}{2} \right) \frac{\partial}{\partial u} + X_F \tag{20}$$

here first term represents non-Hamiltonian part of the generator of the symmetry, while the second one is its Hamiltonian counterpart assosiated with

$$F = \int_{-\infty}^{+\infty} \left(\frac{u^2 v}{12} + \frac{G}{4} + \frac{3vI^{(2)}}{4I^{(3)}} \right) dx \tag{21}$$

($I^{(2,3)}$ are defined in (22), while G is defined by $G_x = \frac{u^3}{3} - u_x^{\ 2}$. The physical origin of this symmetry is unclear, however the symmetry seems to be very important since it leads to the celebrated infinite sequence of conservation laws in involution:

$$I^{(1)} = \int_{-\infty}^{+\infty} u \, dx \tag{22}$$

$$I^{(2)} = \int_{-\infty}^{+\infty} u^2 \, dx$$

$$I^{(3)} = \int_{-\infty}^{+\infty} \left(\frac{u^3}{3} - u_x{}^2 \right) dx$$

$$I^{(4)} = \int_{-\infty}^{+\infty} \left(\frac{5}{36} u^4 - \frac{5}{3} u u_x{}^2 + u_{xx}{}^2 \right) dx$$

$$\ldots$$

and ensures integrability of KdV equation. Second Hamiltonian realization of KdV equation discovered by F. Magri [7]

$$i_{X_{h^*}} \omega^* + dh^* = 0 \tag{23}$$

(where $\omega^* = L_E \omega$ and $h^* = L_E h$) is a result of invariance of KdV under aforementioned transformations g(a).

Acknowledgements. Author is grateful to Z. Giunashvili for constructive discussions and to G. Jorjadze for support. This work was supported by INTAS (00-00561) and Scholarship from World Federation of Scientists.

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